Exam 2 Solutions & Feedback Key

General Feedback Key

I use this key to concisely and consistently mark common comments on your work. In interpreting a mark, please compare your answer to the solutions below. If you'd like further clarification of what I mean in a particular case, you're welcome to drop by office hours.

- AT — **Anything proves true.** Starting with a statement that you want to prove, and then showing that you can derive a true statement from it, does not prove anything about the truth of your original statement.
- CX — This is needlessly **complicated**. Simpler approaches or arguments exist that are equally good, but easier to convey.
- E — Please be more **explicit** here. The labeled work is insufficiently explicit / insufficiently precise.
- IS — The argument you have presented is **insufficient** to prove the conclusion you claim.
- J — There is a problem with your **justification**. "J-" means the justification is incorrect.
- M — Use formal **mathematical** notation here, and/or express what you mean in mathematical terms.
- U — Please submit only one answer to each question.
- UC — This is **unclear** or confusing. Make your reasoning or instructions more clear or straightforward.
- V — There is a problem here with your proof/argument: it is **invalid**, insufficiently rigorous, or otherwise not convincing. This may be because there are holes in your logic, because you made an invalid inference or equivalence, because your conclusions are not supported by your argument, because you have made a significant claim without proof, because you have failed to address an important possibility or case, or for some other reason.

Question 1

(a) $O(n^3)$ (b) $O(2^n)$ (c) $O(n \log n)$ (d) $O(n)$

Question 2

[Each justification in brackets below is just an explanation of the answer, not part of the answer itself.]

(a) $\binom{17}{4} = \frac{17!}{4!(17-4)!}$ *[Justification: order doesn't matter – "soup and salad" is the same meal as "salad and soup" – and repetition's not allowed – that's what "distinct items" means.]*

(b) 8^{15} *[Justification: there are 15 distinguishable positions to fill, and 8 options for each one. In other words, order does matter (President is different from Vice-President) but repetition is allowed.]*

(c) $P(15, 5) = \frac{15!}{(15-5)!}$ *[Justification: we're selecting 5 colors out of a set of 15, and the order of our selections matters – orange text on a black background is different from black text on an orange background.]*

Question 3

Claim: There exist $c > 0, N > 0$ such that $\forall\ n \geq N,\ \frac{1}{2}n^2 - 2 \geq c \cdot n^2$.

[There are two slightly different, but equally valid, approaches to this proof that I think are worth noting.]

- **Proof:** For $n \geq 3, n^2 \geq 9$. So:

  $$\frac{1}{2}n^2 - 2 = \frac{1}{4}n^2 + \frac{1}{4}n^2 - 2 \geq \frac{1}{4}n^2 + \frac{9}{4} - 2 = \frac{1}{4}n^2 + \frac{1}{4} \geq \frac{1}{4}n^2 = c \cdot n^2$$

  Therefore $c = \frac{1}{4}$ and $N = 3$ satisfy the claim.

- **Proof:**

  $$\begin{aligned} & \forall\ n \geq 4\ :\ n^2 \geq 16 \\ \Rightarrow\quad & \forall\ n \geq 4\ :\ \frac{1}{8}n^2 \geq 2 \\ \Rightarrow\quad & \forall\ n \geq 4\ :\ \frac{1}{8}n^2 - 2 \geq 0 \\ \Rightarrow\quad & \forall\ n \geq 4\ :\ \frac{1}{2}n^2 - 2 = \frac{3}{8}n^2 + \frac{1}{8}n^2 - 2 \geq \frac{3}{8}n^2 = c \cdot n^2 \end{aligned}$$

  Therefore $c = \frac{3}{8}$ and $N = 4$ satisfy the claim.

  3.I The statement "$f(n) = \Omega(g(n))$" means that there exist $c > 0, N > 0$ such that for all $n \geq N, f(n) \geq c \cdot g(n)$. Note that N and c must both be strictly positive, that the claim is about $n \geq N$, and that you're aiming to prove an asymptotic *lower* bound on f.

Question 4

$$\begin{aligned} \binom{n}{k-1}\frac{n-k+1}{k} &= \frac{n!}{(k-1)!(n-k+1)!} \cdot \frac{n-k+1}{k} \\ &= \frac{n!(n-k+1)}{k \cdot (k-1)!(n-k+1) \cdot (n-k)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k} \end{aligned}$$

Question 5

(a) 2 (b) Yes (c) 3 (d) 2, 5, 7, 8

Question 6

(a) Yes, 3. (b) Yes, 2. (c) Yes: $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 5 \end{bmatrix}$